UNBOUNDED SUPERSOLUTIONS OF SOME QUASILINEAR PARABOLIC EQUATIONS: A DICHOTOMY

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Abstract. We study unbounded "supersolutions" of the Evolutionary p-Laplace equation with slow diffusion. They are the same functions as the viscosity supersolutions. A fascinating dichotomy prevails: either they are locally summable to the power $p-1+\frac{n}{p}-0$ or not summable to the power $p-2$. There is a void gap between these exponents. Those summable to the power $p-2$ induce a Radon measure, while those of the other kind do not. We also sketch similar results for the Porous Medium Equation.

1. Introduction

The *unbounded* supersolutions of the Evolutionary p-Laplace Equation

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) = 0, \qquad 2 < p < \infty,$$

exhibit a fascinating dichotomy in the slow diffusion case $p > 2$. This phenomenon was discovered and investigated in [14]. The purpose of the present work is to give an alternative proof, directly based on the iterative procedure in [8]. Besides the achieved simplification, our proof can readily be extended to more general quasilinear equations of the form

$$\frac{\partial u}{\partial t} - \nabla \cdot \mathbf{A}(x, t, u, \nabla u) = 0,$$

which are treated in the book [DGV]. The expedient analytic tool is the intrinsic Harnack inequality for positive solutions, see [6]. We can avoid to evoke it for *super*solutions. We also mention the books [4] and [24] as general references.

The supersolutions that we consider are called p-supercaloric functions[1]. They are pointwise defined lower semicontinuous functions, finite in a dense subset, and are required to satisfy the Comparison Principle with respect to the solutions of the equation; see Definition 2.4 below. The definition is the same as the one in classical potential

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[1]They are also called p-parabolic functions, as in [8].

theory for the Heat Equation[2], to which the equation reduces when $p = 2$, see [23]. Incidentally, *the p-supercaloric functions are exactly the viscosity supersolutions of the equation*, see [7].

There are two disjoint classes of p-supercaloric functions, called class $\mathfrak{B}$ and $\mathfrak{M}$. We begin with the former one. Throughout the paper we assume that Ω is an open subset of $\mathbb{R}^n$ and we denote $\Omega_T = \Omega \times (0, T)$ for $T > 0$.

Theorem 1.1 (Class $\mathfrak{B}$). *Let $p > 2$. For a p-supercaloric function $v : \Omega_T \to (-\infty, \infty]$ the following conditions are equivalent:*

(i) $v \in L^{p-2}_{loc}(\Omega_T)$,

(ii) *the Sobolev gradient ∇v exists and $\nabla v \in L^{q'}_{loc}(\Omega_T)$ whenever $q' < p - 1 + \frac{1}{n+1}$,*

(iii) *$v \in L^q_{loc}(\Omega_T)$ whenever $q < p - 1 + \frac{p}{n}$.*

In this case there exists a non-negative Radon measure μ such that

$$\int_0^T \int_\Omega \left(-v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \right) \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} \varphi \, \mathrm{d}\mu \tag{1.2}$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$. In other words, the equation

$$\frac{\partial v}{\partial t} - \nabla \cdot \big(|\nabla v|^{p-2} \nabla v \big) = \mu$$

holds in the sense of distributions, cf. [12]. It is of utmost importance that the local summability exponent for the gradient in (ii) is at least $p - 2$. Such measure data equations have been much studied and we only refer to [2]. For potential estimates we refer to [15], [16].

As an example of a function belonging to class $\mathfrak{B}$ we mention the celebrated Barenblatt solution

$$\mathfrak{B}(x,t) = \begin{cases} t^{-\frac{n}{\lambda}} \left[C - \frac{p-2}{p} \lambda^{\frac{1}{1-p}} \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}, & \text{when} \quad t > 0, \\ 0, \quad \text{when} \quad t \le 0, \end{cases} \tag{1.3}$$

found in 1951, cf. [1]. Here $\lambda = n(p-2) + p$ and $p > 2$. It is a solution of the Evolutionary p-Laplace Equation, except at the origin $x = 0$, $t = 0$. Moreover, it is a p-supercaloric function in the whole $\mathbb{R}^n \times \mathbb{R}$, where it satisfies the equation

$$\frac{\partial \mathfrak{B}}{\partial t} - \nabla \cdot (|\nabla \mathfrak{B}|^{p-2} \nabla \mathfrak{B}) = c\delta$$

in the sense of distributions (δ = Dirac's delta). It also shows that the exponents in (i) and (ii) of the previous theorem are sharp.

[2]Yet, the dichotomy we focus our attention on, is impossible for the Heat Equation.

A very different example is the stationary function

$$v(x,t) = \sum_j \frac{c_j}{|x-q_j|^{\frac{n-p}{p-1}}}, \qquad 2 < p < n,$$

where the q_j's are an enumeration of the rationals and the $c_j \geq 0$ are convergence factors. Indeed, this is a p-supercaloric function, it has a Sobolev gradient, and $v(q_j, t) \equiv \infty$ along every rational line $x = q_j$, $-\infty < t < \infty$, see [18].

Then we describe class $\mathfrak{M}$.

Theorem 1.4 (Class $\mathfrak{M}$)**.** *Let $p > 2$. For a p-supercaloric function $v : \Omega_T \to (-\infty, \infty]$ the following conditions are equivalent:*

(i) $v \notin L^{p-2}_{loc}(\Omega_T)$,

(ii) *there is a time t_0, $0 < t_0 < T$, such that*

$$\liminf_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} v(y,t)(t-t_0)^{\frac{1}{p-2}} > 0 \quad \textit{for all} \quad x \in \Omega.$$

Notice that the infinities occupy the whole space at some instant t_0. As an example of a function from class $\mathfrak{M}$ we mention

$$\mathfrak{V}(x,t) = \begin{cases} \dfrac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}}, & \text{when} \quad t > t_0, \\ 0, \quad \text{when} & t \leq t_0, \end{cases}$$

where $\mathfrak{U} \in C(\Omega) \cap W^{1,p}_0(\Omega)$ is a weak solution to the elliptic equation

$$\nabla \cdot \left(|\nabla \mathfrak{U}|^{p-2} \nabla \mathfrak{U}\right) + \tfrac{1}{p-2}\mathfrak{U} = 0$$

and $\mathfrak{U} > 0$ in Ω. The function $\mathfrak{V}$ is p-supercaloric in $\Omega \times \mathbb{R}$, see equation (3.3) below. This function can serve as a minorant for all functions $v \geq 0$ in $\mathfrak{M}$. No σ-finite measure is induced in this case. As far as we know, these functions have not yet been carefully studied.

A function of class $\mathfrak{M}$ always affects the boundary values. Indeed, at some point on (ξ_0, t_0) on the lateral boundary $\partial\Omega \times (0,T)$ it is necessary to have

$$\limsup_{(x,t)\to(\xi_0,t_0)} v(x,t) = \infty.$$

This alone does not yet prove that v would belong to $\mathfrak{M}$. A convenient sufficient condition for membership in class $\mathfrak{B}$ emerges: *If*

$$\limsup_{(x,\tau)\to(\xi,t)} v(x,\tau) < \infty \quad \textit{for every} \quad (\xi,t) \in \partial\Omega \times (0,T),$$

then $v \in \mathfrak{B}$.

It is no surprise that a parallel theory holds for the celebrated Porous Medium Equation

$$\frac{\partial u}{\partial t} - \Delta(u^m) = 0, \qquad 1 < m < \infty.$$

We refer to the monograph [22] about this much studied equation. We sketch the argument in the last section.

2. Preliminaries

We begin with some standard notation. We consider an open domain Ω in $\mathbb{R}^n$ and denote by $L^p(t_1,t_2;W^{1,p}(\Omega))$ the Sobolev space of functions $v = v(x,t)$ such that for almost every $t, t_1 \le t \le t_2$, the function $x \mapsto v(x,t)$ belongs to $W^{1,p}(\Omega)$ and

$$\int_{t_1}^{t_2}\int_\Omega \left(|v(x,t)|^p + |\nabla v(x,t)|^p\right)\,\mathrm{d}x\,\mathrm{d}t \;<\; \infty,$$

where $\nabla v = (\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n})$ is the spatial Sobolev gradient. The definitions of the local spaces $L^p(t_1,t_2;W^{1,p}_{loc}(\Omega))$ and $L^p_{loc}(t_1,t_2;W^{1,p}_{loc}(\Omega))$ are analogous. We denote $\Omega_{t_1,t_2} = \Omega \times (t_1,t_2)$ and recall that the *parabolic boundary* of Ω_{t_1,t_2} is the set $\big(\overline{\Omega}\times\{t_1\}\big) \cup \big(\partial\Omega \times (t_1,t_2)\big)$.

Definition 2.1. A function $u \in L^p(t_1,t_2;W^{1,p}(\Omega))$ is a *weak solution* of the Evolutionary p-Laplace Equation in Ω_{t_1,t_2}, if

$$\int_{t_1}^{t_2}\int_\Omega \left(-u\frac{\partial\varphi}{\partial t} + \langle|\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\right)\mathrm{d}x\,\mathrm{d}t = 0 \tag{2.2}$$

for every $\varphi \in C_0^\infty(\Omega_{t_1,t_2})$. If, in addition, u is continuous, then it is called a *p-caloric function.* Further, we say that u is a *weak supersolution*, if the above integral is non-negative for all non-negative $\varphi \in C_0^\infty(\Omega_{t_1,t_2})$. If the integral is non-positive instead, we say that u is a *weak subsolution.*

By parabolic regularity theory, a weak solution is locally Hölder continuous after a possible redefinition in a set of $n+1$-dimensional Lebesgue measure zero, see [21] and [4]. In addition, a weak supersolution is upper semicontinuous with the same interpretation, cf. [13].

Lemma 2.3 (Comparison Principle)**.** *Assume that*

$$u,\, v \in L^p\big(t_1,t_2;W^{1,p}(\Omega)\big) \cap C\big(\overline{\Omega}\times[t_1,t_2)\big).$$

If v is a weak supersolution and u a weak subsolution in Ω_{t_1,t_2} such that $v \ge u$ on the parabolic boundary of Ω_{t_1,t_2}, then $v \ge u$ in the whole Ω_{t_1,t_2}.

The Comparison Principle is used to define the class of p-supercaloric functions.

Definition 2.4. A function $v : \Omega_{t_1,t_2} \to (-\infty,\infty]$ is called *p-supercaloric*, if

(i) v is lower semicontinuous,
(ii) v is finite in a dense subset,

(iii) v satisfies the comparison principle on each interior cylinder $D_{t_1',t_2'} \Subset \Omega_{t_1,t_2}$: If $h \in C(\overline{D_{t_1',t_2'}})$ is a p-parabolic function in $D_{t_1',t_2'}$, and if $h \leq v$ on the parabolic boundary of $D_{t_1',t_2'}$, then $h \leq v$ in the whole $D_{t_1',t_2'}$.

We recall a fundamental result for *bounded* functions, which is also applicable to more general equations.

Theorem 2.5. *Let $p \geq 2$. If v is a p-supercaloric function that is locally bounded from above in Ω_T, then the Sobolev gradient ∇v exists and $\nabla v \in L^p_{loc}(\Omega_T)$. Moreover, $v \in L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega))$ and v is a weak supersolution.*

A proof based on auxiliary obstacle problems was given in [9], Theorem 1.4. A more direct proof with infimal convolutions can be found in [17].

In order to apply the previous theorem, we need *bounded* functions. The truncations

$$v_j(x,t) = \min\{v(x,t), j\}, \qquad j = 1, 2, \dots,$$

are p-supercaloric, if v is, and since they are bounded from above, they are also weak supersolutions. Thus ∇v_j is at our disposal and estimates derived from the inequality

$$\int_0^T \int_\Omega \left(-v_j \frac{\partial \varphi}{\partial t} + \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \varphi \rangle \right) \mathrm{d}x \, \mathrm{d}t \geq 0, \tag{2.6}$$

where $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega_T)$, are available. The starting point for our proof is the following theorem for the truncated functions.

Theorem 2.7. *Let $p > 2$. Suppose that $v \geq 0$ is a p-supercaloric function in Ω_T with initial values $v(x,0) = 0$ in Ω. If $v_j \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $j = 1, 2, \dots,$ then*

(i) *$v \in L^q(\Omega_{T_1})$ whenever $q < p - 1 + \frac{p}{n}$ and $T_1 < T$,*
(ii) *the Sobolev gradient ∇u exists and $\nabla v \in L^{q'}(\Omega_{T_1})$ whenever $q' < p - 1 + \frac{1}{n+1}$ and $T_1 < T$.*

Proof. See [9]. □

We remark that the summability exponents are sharp. It is decisive that the boundary values are zero. The functions of class $\mathfrak{M}$ cannot satisfy this requirement. As we shall see, those of class $\mathfrak{B}$ can be modified so that the theorem above applies.

The standard Caccioppoli estimates are valid. We recall the following simple version, which will suffice for us.

Lemma 2.8 (Caccioppoli)**.** *Let $p > 2$. If $u \geq 0$ is a weak subsolution in Ω_T, then the estimate*

$$\int_{t_1}^{t_2} \int_\Omega \zeta^p |\nabla u|^p \, \mathrm{d}x \, \mathrm{d}t + \operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega \zeta(x)^p u(x,t)^2 \, \mathrm{d}x \\ \leq C(p) \left\{ \int_{t_1}^{t_2} \int_\Omega u^p |\nabla \zeta|^p \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \zeta(x)^p u(x,t)^2 \Big|_{t_1}^{t_2} \mathrm{d}x \right\}$$

holds for every $\zeta = \zeta(x) \geq 0$ in $C_0^\infty(\Omega)$, $0 < t_1 < t_2 < T$.

Proof. A formal calculation with the test function $\phi = v\zeta^p$ gives the inequality. See [4], [9]. □

Infimal Convolutions. The infimal convolutions preserve the p-supercaloric functions and are Lipschitz continuous. Thus they are convenient approximations. If $v \geq 0$ is lower semicontinuous and finite in a dense subset of Ω_T, then the *infimal convolution*

$$v^\varepsilon(x,t) = \inf_{(y,\tau)\in\Omega_T} \Big\{ v(y,\tau) + \frac{1}{2\varepsilon}\big(|x-y|^2 + |t-\tau|^2\big) \Big\}$$

is well defined. It has the properties

- $v^\varepsilon(x,t) \nearrow v(x,t)$ as $\varepsilon \to 0$,
- v^ε is locally Lipschitz continuous in Ω_T,
- the Sobolev derivatives $\frac{\partial v^\varepsilon}{\partial t}$ and ∇v^ε exist and belong to $L^\infty_{loc}(\Omega_T)$.

Assume now that v is a p-supercaloric function in Ω_T. Given a subdomain $D \Subset \Omega_T$, the above v^ε is a p-supercaloric function in D, provided that ε is small enough, see [9].

3. A Separable Minorant

We begin with observations, which will simplify some arguments later.

Extension to the past. If v is a non-negative p-supercaloric function in Ω_T, then the extended function

$$v(x,t) = \begin{cases} v(x,t), & \text{when} \quad 0 < t < T, \\ 0, & \text{when} \quad t \leq 0, \end{cases}$$

is p-supercaloric in $\Omega \times (-\infty, T)$. We use the same notation for the extended function.

A separable minorant. Separation of variables suggests that there are p-caloric functions of the type

$$v(x,t) = (t-t_0)^{-\frac{1}{p-2}} u(x).$$

Indeed, if Ω is a domain of finite measure, there exists a p-caloric function of the form

$$\mathfrak{V}(x,t) = \frac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}}, \quad \text{when} \quad t > t_0, \tag{3.1}$$

where $\mathfrak{U} \in C(\Omega) \cap W_0^{1,p}(\Omega)$ is a weak solution to the elliptic equation

$$\nabla \cdot \big(|\nabla \mathfrak{U}|^{p-2} \nabla \mathfrak{U}\big) + \tfrac{1}{p-2}\mathfrak{U} = 0 \tag{3.2}$$

and $\mathfrak{U} > 0$ in Ω. The solution $\mathfrak{U}$ is unique[3]. (Actually, $\mathfrak{U} \in C^{1,\alpha}_{loc}(\Omega)$ for some exponent $\alpha = \alpha(n,p) > 0$.) The extended function

$$\mathfrak{V}(x,t) = \begin{cases} \dfrac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}}, & \text{when} \quad t > t_0, \\ 0, \quad \text{when} \quad t \le t_0. \end{cases} \tag{3.3}$$

is p-supercaloric in $\Omega \times \mathbb{R}$. The existence of $\mathfrak{U}$ follows by the direct method in the Calculus of Variations, when the quotient

$$J(w) = \frac{\int_\Omega |\nabla w|^p \,\mathrm{d}x}{\Big(\int_\Omega w^2 \,\mathrm{d}x\Big)^{\frac{p}{2}}}$$

is minimized among all functions w in $W_0^{1,p}(\Omega)$ with $w \not\equiv 0$. Replacing w by its absolute value $|w|$, we may assume that all functions are non-negative. Sobolev's and Hölder's inequalities imply

$$J(w) \ge c(p,n)|\Omega|^{1-\frac{p}{n}-\frac{p}{2}},$$

for some $c(p,n) > 0$ and so $J_0 = \inf_w J(w) > 0$. Choose a minimizing sequence of admissible normalized functions w_j with

$$\lim_{j\to\infty} J(w_j) = J_0 \quad \text{and} \quad \|w_j\|_{L^p(\Omega)} = 1.$$

By compactness, we may extract a subsequence such that $\nabla w_{j_k} \rightharpoonup \nabla w$ weakly in $L^p(\Omega)$ and $w_{j_k} \to w$ strongly in $L^p(\Omega)$ for some function w. The weak lower semicontinuity of the integral implies that

$$J(w) \le \liminf_{k\to\infty} J(w_{j_k}) = J_0.$$

Since $w \in W_0^{1,p}(\Omega)$ this means that w is a minimizer. We have $w \ge 0$, and $w \not\equiv 0$ because of the normalization.

It follows that w has to be a weak solution of the Euler–Lagrange equation

$$\nabla \cdot \big(|\nabla w|^{p-2} \nabla w\big) + J_0 \|w\|^{p-2}_{L^p(\Omega)} w = 0$$

with $\|w\|_{L^p(\Omega)} = 1$. By elliptic regularity theory $w \in C(\Omega)$, see [20]. Finally, since $\nabla \cdot \big(|\nabla w|^{p-2}\nabla w\big) \le 0$ in the weak sense and $w \ge 0$

[3] Unfortunately, the otherwise reliable paper [J. Garci'a Azorero, I. Peral Alonso: *Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues*, Communications in Partial Differential Equations **12**, 1987, pp. 1389–1430], contains a misprint exactly for those parameter values that would yield this function.

we have that $w > 0$ by the Harnack inequality [20]. A normalization remains to be done. The function

$$\mathfrak{U} = Cu, \quad \text{where} \quad J_0 C^{p-2} = \tfrac{1}{p-2},$$

will do.

One dimensional case. In one dimension the equation is

$$\frac{d}{dx}\Big(|\mathfrak{U}'|^{p-2}\mathfrak{U}'\Big) + \frac{1}{p-2}\mathfrak{U} = 0, \qquad 0 \le x \le L.$$

It has the first integral

$$\frac{p-1}{p}|\mathfrak{U}'|^p + \frac{\mathfrak{U}^2}{2(p-1)} = C$$

in the interval $[0, L]$. Now $\mathfrak{U}(0) = 0 = \mathfrak{U}(L)$ and $\mathfrak{U}'(\frac{L}{2}) = 0$. This determines the constant of integration in terms of $\mathfrak{U}'(0)$ or of the maximal value $M = \max \mathfrak{U} = \mathfrak{U}(\frac{L}{2})$. Solving for $\mathfrak{U}'$, separating the variables, and integrating from 0 to $\frac{L}{2}$, one easily obtains the parameters

$$M = C_1(p) L^{\frac{p}{p-2}} \quad \text{and} \quad \mathfrak{U}'(0) = -\mathfrak{U}'(L) = C_2(p) L^{\frac{2}{p-2}}.$$

The constants can be evaluated. In passing, we mention that $\frac{\mathfrak{U}(x)}{M}$ has interesting properties as a special function.

4. Harnack's Convergence Theorem

A known phenomenon for an increasing sequence of non-negative p-caloric functions is described in this section. The analytic tool is an intrinsic version of Harnack's inequality, see [?], pp. 157–158, [5], and [6]

Lemma 4.1 (Harnack's inequality)**.** *Let $p > 2$. There are constants C and γ, depending only on n and p, such that if $u > 0$ is a lower semicontinuos weak solution in*

$$B(x_0, 4R) \times (t_0 - 4\theta, t_0 + 4\theta), \quad \textit{where} \quad \theta = \frac{CR^p}{u(x_0, t_0)^{p-2}},$$

then the inequality

$$u(x_0, t_0) \le \gamma \inf_{B_R(x_0)} u(x, t_0 + \theta) \tag{4.2}$$

is valid.

Notice that the waiting time θ depends on the solution itself.

Proposition 4.3. *Suppose that we have an increasing sequence $0 \le h_1 \le h_2 \le h_3 \le \dots$ of p-caloric functions in Ω_T and denote $h = \lim_{k\to\infty} h_k$. If there is a sequence $(x_k, t_k) \to (x_0, t_0)$ such that $h_k(x_k, t_k) \to +\infty$, where $x_0 \in \Omega$ and $0 < t_0 < T$, then*

$$\liminf_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} h(y,t)(t - t_0)^{\frac{1}{p-2}} > 0 \quad \textit{for all} \quad x \in \Omega.$$

Thus, at time t_0,

$$\lim_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} h(y,t) \equiv \infty \quad \textit{in} \quad \Omega.$$

Remark 4.4. The limit function h may be finite at every point, though locally unbounded. Keep the function $\mathfrak{V}$ in mind. — The proof will give

$$h(x,t) \geq \frac{\mathfrak{U}(x)}{(t-t_0)^{\frac{1}{p-2}}} \quad \text{in} \quad \Omega \times (t_0, T).$$

Proof: Let $B(x_0, 4R) \Subset \Omega$. Since

$$\theta_k = \frac{CR^p}{h_k(x_k,t_k)^{p-2}} \to 0,$$

Harnack's Inequality (4.2) implies

$$h_k(x_k,t_k) \leq \gamma h_k(x, t_k + \theta_k) \tag{4.5}$$

when $x \in B(x_k, R)$ provided $B(x_k, 4R) \times (t_k - 4\theta_k, t_k + 4\theta_k) \Subset \Omega_T$. The center is moving, but since $x_k \to x_0$, equation (4.5) holds for sufficiently large indices. Let $\Lambda > 1$. We want to compare the solutions

$$\frac{\mathfrak{U}^{\mathfrak{R}}(x)}{\big(t - t_k + (\Lambda-1)\theta_k\big)^{\frac{1}{p-2}}} \quad \text{and} \quad h_k(x,t)$$

when $t = t_k + \theta_k$ and $x \in B(x_0, R)$. Here $\mathfrak{U}^{\mathfrak{R}}$ is the positive solution of the elliptic equation (3.2) in $B(x_0, R)$ with boundary values zero. We get

$$\begin{aligned}
\frac{\mathfrak{U}^{\mathfrak{R}}(x)}{\big(t - t_k + (\Lambda-1)\theta_k\big)^{\frac{1}{p-2}}}\Bigg|_{t=t_k+\theta_k} &= \frac{\mathfrak{U}^{\mathfrak{R}}(x)}{(\Lambda C R^p)^{\frac{1}{p-2}}} h_k(x_k,t_k) \\
&\leq \frac{\mathfrak{U}^{\mathfrak{R}}(x)}{(\Lambda C R^p)^{\frac{1}{p-2}}} \gamma h_k(x, t_k+\theta_k) \leq h_k(x, t_k + \theta_k)
\end{aligned}$$

by taking Λ so large that

$$\frac{\gamma \|\mathfrak{U}^{\mathfrak{R}}\|_{L^\infty(B(x_0,R))}}{(\Lambda C R^p)^{\frac{1}{p-2}}} \leq 1.$$

By the Comparison Principle

$$\frac{\mathfrak{U}^{\mathfrak{R}}(x)}{\big(t - t_k + (\Lambda-1)\theta_k\big)^{\frac{1}{p-2}}} \leq h_k(x,t) \leq h(x,t)$$

when $t \geq t_k + \theta_k$ and $x \in B(x_0, R)$. By letting $k \to \infty$, we arrive at

$$\frac{\mathfrak{U}^{\mathfrak{R}}(x)}{\big(t - t_0\big)^{\frac{1}{p-2}}} \leq h(x,t) \quad \text{when} \quad t_0 < t < T.$$

Here $\mathfrak{U}^{\mathfrak{R}}$ depended on the ball $B(x_0, R)$, but now we have many more infinities, so that we may repeat the procedure in a suitable chain of balls to extend the estimate to the whole domain Ω. $\square$

Proposition 4.6. *Suppose that we have an increasing sequence $0 \leq h_1 \leq h_2 \leq h_3 \leq \dots$ of p-caloric functions in Ω_T and denote $h = \lim_{k\to\infty} h_k$. If the sequence $\{h_k\}$ is locally bounded, then the limit function h is p-caloric in Ω_T.*

Proof. In a strict subdomain we have the Hölder continuity estimate

$$|h_k(x_1,t_1) - h_k(x_2,t_2)| \leq C\|h_k\| \left(|x_2 - x_1|^{\alpha} + |t_2 - t_1|^{\frac{\alpha}{p}}\right)$$

so that the family is locally equicontinuous. Hence the convergence $h_k \to h$ is locally uniform in Ω_T. Theorem 24 in [LM] implies that $\{\nabla h_k\}$ is a Cauchy sequence in $L^{p-1}_{loc}(\Omega_T)$. Thus we can pass to the limit under the integral sign in the equation

$$\int_0^T \int_\Omega \left(-h_k \frac{\partial \varphi}{\partial t} + \langle |\nabla h_k|^{p-2} \nabla h_k, \nabla \varphi \rangle\right) \mathrm{d}x\, \mathrm{d}t = 0$$

as $k \to \infty$. From the Caccioppoli estimate

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_\Omega \zeta^p |\nabla h_k|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C(p) \int_{t_1}^{t_2} \int_\Omega h_k^p |\nabla \zeta|^p \,\mathrm{d}x\,\mathrm{d}t + C(p) \int_\Omega \zeta(x)^p h_k(x,t)^2 \Big|_{t_1}^{t_2} \mathrm{d}x
\end{aligned}$$

we deduce that $h \in L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega))$. $\square$

5. Proof of the Theorem

For the proof we start with a non-negative p-supercaloric function v defined in Ω_T. By the device in the beginning of Section 3, we fix a small $\delta > 0$ and redefine v so that $v(x,t) \equiv 0$ when $t \leq \delta$. This function is p-supercaloric. This does not affect the statement of the theorem. The initial condition $v(x,0) = 0$ required in Theorem 2.7 is now in order.

Let $Q_{2l} \subset\subset \Omega$ be a cube with side length $4l$ and consider the concentric cube

$$Q_l = \{x \big| |x_i - x_i^0| < l, i = 1, 2, \dots n\}$$

of side length $2l$. The center is at x^0. The main difficulty is that v is not zero on the lateral boundary, neither does v_j obey Theorem 2.7. We aim at correcting v outside $Q_l \times (0,T)$ so that also the new function is p-supercaloric and, in addition, satisfies the requirements of zero boundary values in Theorem 2.7. Thus we study the function

$$w = \begin{cases} v & \text{in} \quad Q_l \times (0,T), \\ h & \text{in} \quad (Q_{2l} \setminus Q_l) \times (0,T), \end{cases} \tag{5.1}$$

where the function h is, in the outer region, the weak solution to the boundary value problem

$$\begin{cases} h = 0 & \text{on} \quad \partial Q_{2l} \times (0,T), \\ h = v & \text{on} \quad \partial Q_l \times (0,T), \\ h = 0 & \text{on} \quad (Q_{2l} \setminus Q_l) \times \{0\}. \end{cases} \tag{5.2}$$

An essential observation is that the solution h does not always exist. This counts for the dichotomy. If it exists, the truncations w_j satisfy the assumptions in Theorem 2.7, as we shall see.

For the construction we use the infimal convolutions

$$v^\varepsilon(x,t) = \inf_{(y,\tau)\in\Omega_T} \Big\{ v(y) + \frac{1}{2\varepsilon}(|x-y|^2 + |t-\tau|^2) \Big\}.$$

They are Lipschitz continuous in $\overline{Q_{2l}} \times [0,T]$ and weak supersolutions when ε is small enough. Then we define the solution h^ε as in formula (5.2) above, but with v^ε in place of v. Then we define

$$w^\varepsilon = \begin{cases} v^\varepsilon & \text{in} \quad Q_l \times (0,T), \\ h^\varepsilon & \text{in} \quad (Q_{2l} \setminus Q_l) \times (0,T), \end{cases}$$

and $w^\varepsilon(x,0) = 0$ in Ω. Now $h^\varepsilon \le v^\varepsilon$, and when $t \le \delta$ we have $0 \le h^\varepsilon \le v^\varepsilon = 0$ so that $h^\varepsilon(x,t) = 0$ when $t \le \delta$. The function w^ε satisfies the comparison principle and is therefore a p-supercaloric function. Here it is essential that $h^\varepsilon \le v^\varepsilon$. The function w^ε is also (locally) bounded; thus we have arrived at the conclusion that w^ε is a weak supersolution in $Q_{2l} \times (0,T)$.

There are two possibilities, depending on whether the sequence $\{h^\varepsilon\}$ is bounded or not, when $\varepsilon \searrow 0$ through a sequence of values.

Bounded case. Assume that there does not exist any sequence of points $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ such that

$$\lim_{\varepsilon\to 0} h^\varepsilon(x_\varepsilon, t_\varepsilon) = \infty,$$

where $x_0 \in Q_{2l} \setminus \overline{Q_l}$ and $0 < t_0 < T$ (that is an *interior* limit point). By Proposition 4.6, the limit function $h = \lim_{\varepsilon\to 0} h^\varepsilon$ is p-caloric in its domain. The function $w = \lim_{\varepsilon\to 0} w^\varepsilon$ itself is p-supercaloric and agrees with formula (5.1).

By Theorem 2.5 the truncated functions $w_j = \min\{w(x,t), j\}$, $j = 1, 2, \ldots$, are weak supersolutions in $Q_{2l} \times (0,T)$. We claim that

$$w_j \in L^p(0,T'; W_0^{1,p}(Q_{2l})) \quad \text{when} \quad T' < T.$$

This requires an estimation, where we use

$$L = \sup\{h(x,t) : (x,t) \in (Q_{2l} \setminus Q_{5l/4}) \times (0,T')\}.$$

Let $\zeta = \zeta(x)$ be a smooth cutoff function such that $0 \le \zeta \le 1$, $\zeta = 1$ in $Q_{2l} \setminus Q_{3l/2}$ and $\zeta = 0$ in $Q_{5l/4}$. Using the test function $\zeta^p h$ when

deriving the Caccioppoli estimate we get

$$\begin{aligned}
&\int_0^{T'}\int_{Q_{2l}\setminus Q_{3l/2}} |\nabla w_j|^p \,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^{T'}\int_{Q_{2l}\setminus Q_{3l/2}} |\nabla h|^p \,\mathrm{d}x\,\mathrm{d}t \le \int_0^{T'}\int_{Q_{2l}\setminus Q_{5l/4}} \zeta^p|\nabla h|^p \,\mathrm{d}x\,\mathrm{d}t\\
&\le C(p)\left\{\int_0^{T'}\int_{Q_{2l}\setminus Q_l} h^p|\nabla \zeta|^p \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{2l}\setminus Q_{5l/4}} h(x,T')^2 \,\mathrm{d}x\right\}\\
&\le C(n,p)\Big(L^p l^{n-p}T + L^2 l^n\Big),
\end{aligned}$$

where we used the fact that $|\nabla w_j| = |\nabla \min\{h,j\}| \le |\nabla h|$ in the outer region. Thus we have an estimate over the outer region $Q_{2l}\setminus Q_{3l/2}$. Concerning the inner region $Q_{3l/2}$, we first choose a smooth cutoff function $\eta = \eta(x,t)$ such that $0\le\eta\le 1$, $\eta\equiv 1$ in $Q_{3l/2}$ and $\eta = 0$ in $Q_{2l}\setminus Q_{9l/4}$. Then the Caccioppoli estimate for the truncated functions w_j, $j = 1,2,\dots$, takes the form

$$\begin{aligned}
&\int_0^{T'}\int_{Q_{3l/2}} |\nabla w_j|^p \,\mathrm{d}x\,\mathrm{d}t \le \int_0^{T'}\int_{Q_{2l}} \eta^p|\nabla w_j|^p \,\mathrm{d}x\,\mathrm{d}t\\
&\le Cj^p\int_0^{T'}\int_{Q_{2l}} |\nabla \eta|^p \,\mathrm{d}x\,\mathrm{d}t + Cj^p\int_0^{T'}\int_{Q_{2l}} |\eta_t|^p \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Thus we have obtained the estimate

$$\int_0^{T'}\int_{Q_{2l}} |\nabla w_j|^p \,\mathrm{d}x\,\mathrm{d}t \le Cj^p$$

over the whole domain $Q_{2l}\times(0,T')$ and it follows that $w_j \in L(0,T';W_0^{1,p}(Q_{2l}))$. In particular, the crucial estimate

$$\int_0^{T'}\int_{Q_{2l}} |\nabla w_1|^p \,\mathrm{d}x\,\mathrm{d}t < \infty,$$

which was taken for granted in [10], is now established.[4]

From Theorem 2.7 we conclude that $v\in L^q(Q_l)$ and $\nabla v \in L^{q'}(Q_l)$ with the correct summability exponents. Either we can proceed like this for all interor cubes, or the following case occurs.

Unbounded case. If there is a sequence $(x_\varepsilon,t_\varepsilon)\to(x_0,t_0)$ such that

$$\lim_{\varepsilon\to 0} h^\varepsilon(x_\varepsilon,t_\varepsilon) = \infty$$

for some $x_0\in Q_{2l}\setminus\overline{Q_l}$, $0<t_0<T$, then

$$v(x,t)\ge h(x,t)\ge (t-t_0)^{-\frac{1}{p-2}}\mathfrak{U}(x),$$

[4]The class $\mathfrak{M}$ passed unnoticed in [10].

when $t > t_0$, according to Proposition 4.3. Thus $v(x, t_0+) = \infty$ in $Q_{2l} \setminus \overline{Q_l}$. But in this construction we can replace the outer cube with Ω, that is, a new h is defined in $\Omega \setminus \overline{Q_l}$. Then by comparison

$$v \geq h^{\Omega} \geq h^{Q_{2l}}$$

and so $v(x, t_0+) = \infty$ in the whole boundary zone $\Omega \setminus \overline{Q_l}$.

It remains to include the inner cube Q_l in the argument. This is easy. Reflect $h = h^{Q_{2l}}$ in the plane $x_1 = x_1^0 + l$, which contains one side of the small cube by setting

$$h^*(x_1, x_2 \dots, x_n) = h(2x_1^0 + 2l - x_1, x_2, \dots, x_n),$$

so that

$$\frac{x_1 + \big(2(x_1^0 + l) - x_1\big)}{2} = x_1^0 + l$$

as it should. Recall that x^0 was the center of the cube. (The same can be done earlier for all the h^{ε}.) The reflected function h^* is p-caloric. Clearly, $v \geq h^*$ by comparison. This forces $v(x, t_0+) = 0$ when $x \in Q_l$, $x_1 > x_1^0$. A similar reflexion in the plane $x_1 = x_1^0 - l$ includes the other half $x_1 < x_1^0$. We have achieved that $v(x, t_0+) = \infty$ also in the inner cube Q_l. This proves that

$$v(x, t_0+) \equiv \infty \quad \text{in the whole} \quad \Omega.$$

6. The Porous Medium Equation

We consider the Porous Medium Equation

$$\frac{\partial u}{\partial t} - \Delta(u^m) = 0$$

in the slow diffusion case $m > 1$. The equation is reated in detail in the book [22]. We also mention [24] and [19]. In [11] the so-called[5] *viscosity supersolutions* of the Porous Medium Equation were defined in an analoguous way as the p-supercaloric functions. Thus they are lower semicontinuous functions $v : \Omega_T \to [0, \infty]$, finite in a dense subset, obeying the Comparison Principle with respect to the solutions of the equation.

Again we get two totally distinct classes of solutions, called class $\mathfrak{B}$ and $\mathfrak{M}$. Now the discriminating summability exponent is $m - 1$. We begin with $\mathfrak{B}$.

Theorem 6.1 (Class $\mathfrak{B}$). *Let $m > 1$. For a viscosity supersolution $v : \Omega_T \to [0, \infty]$ the following conditions are equivalent:*

(i) *$v \in L^{m-1}_{loc}(\Omega_T)$,*

(ii) *the Sobolev gradient $\nabla(v^{m-1})$ exists and $\nabla(v^{m-1}) \in L^{q'}_{loc}(\Omega_T)$ whenever $q' < 1 + \frac{1}{1+nm}$,*

[5]The label "viscosity" was dubbed in order to distinguish them and has little to do with viscosity. The name "m-superporous function" would perhaps do instead?

(iii) $v \in L^q_{loc}(\Omega_T)$ *whenever* $q < m + \frac{2}{n}$.

A typical member of this class is the Barenblatt solution for the Porous Medium Equation. In this case a viscosity supersolution is a solution to a corresponding measure data problem with a Radon measure in a similar fashion as for the Evolutionary p-Laplace Equation. The other class of viscosity supersolutions is $\mathfrak{M}$. Unfortunately, this class was overlooked in [11].

Theorem 6.2 (Class $\mathfrak{M}$)**.** *Let* $m > 1$. *For a viscosity supersolution* $v : \Omega_T \to [0, \infty]$ *the following conditions are equivalent:*

(i) $v \notin L^{m-1}_{loc}(\Omega_T)$,

(ii) *there is a time* t_0, $0 < t_0 < T$, *such that*

$$\liminf_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} v(y,t)(t-t_0)^{\frac{1}{m-1}} > 0 \quad \textit{for all} \quad x \in \Omega.$$

Notice that again the infinities occupy the whole space at some instant t_0. In [11] Theorem 3.2 it was established that a *bounded* viscosity supersolution v is a weak supersolution to the equation: $v^m \in L^2_{loc}(0, t; W^{1,2}_{loc}(\Omega))$ and

$$\int_0^T \int_\Omega \left(-v \frac{\partial \phi}{\partial t} + \langle \nabla v^m, \nabla \varphi \rangle \right) \mathrm{d}x \, \mathrm{d}t \geq 0$$

whenever $\varphi \in C_0^\infty(\Omega_T)$ and $\varphi \geq 0$.

We shall deduce the above theorems from the following result.

Theorem 6.3. *Let* $m > 1$. *Suppose that* $v \geq 0$ *is a viscosity supersolution in* Ω_T *with initial values* $v(x, 0) = 0$ *in* Ω. *If*

$$\min\{v^m, j\} \in L^2(0, T; W_0^{1,2}(\Omega)), \qquad j = 1, 2, \ldots,$$

then

(i) $v \in L^q(\Omega_{T_1})$ *whenever* $q < 1 + \frac{2}{n}$ *and* $T_1 < T$,

(ii) *the function* v^m *has a Sobolev gradient* $\nabla(v^m) \in L^{q'}(\Omega_{T_1})$ *whenever* $q' < 1 + \frac{1}{1+mn}$ *and* $T_1 < T$.

The summability exponents are sharp.

Proof. See [11], Theorem 4.7 and 4.8. □

We start from the intrinsic Harnack inequality given in [3, Theorem 3]. This is the fundamental analytic tool here.

Lemma 6.4 (Harnack's inequalty)**.** *Let* $m > 1$. *There are constants* C *and* γ, *depending only on* n *and* m, *such that if* $u > 0$ *is a continuos weak solution in*

$$B(x_0, 4R) \times (t_0 - 4\theta, t_0 + 4\theta), \quad \textit{where} \quad \theta = \frac{CR^2}{u(x_0, t_0)^{m-1}},$$

then the inequality

$$u(x_0, t_0) \le \gamma \inf_{B_R(x_0)} u(x, t_0 + \theta) \tag{6.5}$$

is valid.

Again the waiting time θ depends on the solution itself. Then we need the separable solution

$$\frac{\mathfrak{G}(x)}{(t - t_0)^{\frac{1}{m-1}}}$$

where the function $\mathfrak{G}^m \in W_0^{1,2}(\Omega)$ is a weak solution of the auxiliary equation

$$\Delta(\mathfrak{G}^m) + \frac{\mathfrak{G}}{m-1} = 0,$$

which is the Euler-Lagrange Equation of the variational integral

$$\frac{\int_\Omega |\nabla(u^m)|^2 \, \mathrm{d}x}{\int_\Omega |u|^{m+1} \, \mathrm{d}x}.$$

This function is known as "the Friendly Giant", see [V, p. 111] and often serves as a minorant. When extended as 0 when $t < t_0$ it becomes a viscosity supersolution in the whole $\Omega \times \mathbb{R}$.

Proposition 6.6. *Suppose that we have an increasing sequence* $0 \le h_1 \le h_2 \le h_3 \le \dots$ *of viscosity supersolutions in* Ω_T *and denote* $h = \lim_{k\to\infty} h_k$*. If there is a sequence* $(x_k, t_k) \to (x_0, t_0)$ *such that* $h_k(x_k, t_k) \to \infty$*, where* $x_0 \in \Omega$ *and* $0 < t_0 < T$*, then*

$$\liminf_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} h(y,t)(t - t_0)^{\frac{1}{m-1}} > 0 \quad \textit{for all} \quad x \in \Omega.$$

Thus, at time t_0*,*

$$\lim_{\substack{(y,t)\to(x,t_0)\\ t>t_0}} h(y,t) \equiv \infty \quad \textit{in} \quad \Omega.$$

Remark 6.7. Notice that the limit function is not a solution in the whole domain, but it may, nonetheless, be finite at each point. (This is different from the Heat Equation, see [23].)

If it so happens that the subsequence in the Proposition does not exist, then we have the normal situation with a solution:

Proposition 6.8. *Suppose that we have an increasing sequence* $0 \le h_1 \le h_2 \le h_3 \le \dots$ *of viscosity supersolutions in* Ω_T *and denote* $h = \lim_{k\to\infty} h_k$*. If the sequence* $\{h_k\}$ *is locally bounded, then the limit function* h *is a supersolution in* Ω_T*.*

After this, the proof proceeds along the same lines as for the p-parabolic equation. A difference is that the infimal convolution should be replaced by the solution to an obstacle problem as in Chapter 5 of [11].

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